A note on sums of five almost equal prime squares

By

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Abstract. Let $N$ be any sufficiently large positive integer satisfying the congruence condition $N \equiv 5 \pmod{24}$. It is shown that there exists a $\delta > 0$ such that $N$ can be written as

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,$$

where the $p_i$ are prime numbers and $U$ is chosen as $U = N^{\frac{1}{5}}$.  

1. Introduction and statement of results. One of Hua’s outstanding contributions to prime number theory was to prove that every sufficiently large integer $N \equiv 5 \pmod{24}$ can be written as the sum of five prime squares ([3]). Recently, Liu and Zhan ([6]) were able to sharpen this result in the following way:

Theorem 1. Assume the Great Riemann Hypothesis. Denote by $R(N, U)$ the number of solutions of the Diophantine equation with prime variables

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,$$

where

$$p_j - \sqrt{\frac{N}{5}} \leq U, \quad j = 1, 2, 3, 4, 5,$$

Then for $U = N^{\frac{2}{5}+\varepsilon}$, we have

$$R(N, U) = \frac{460\sqrt{5}}{3} \sigma(N) \frac{U^{4}}{N^{4 \log^{2} N}} (1 + o(1)),$$

where

$$\sigma(N) = \sum_{q=1}^{\infty} \frac{1}{q^{3}(n)} \sum_{\substack{a=1 \\text{mod} q \atop (a, q) = 1}}^{q} C^5(a, q) e\left(-\frac{aN}{q}\right)$$

with

$$C(a, q) = \sum_{\substack{h=1 \\text{mod} q \atop (h, a) = 1}}^{q} e\left(\frac{ah^2}{q}\right).$$

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Here $\sigma(N)$ is the so-called singular series, which is convergent and satisfies $\sigma(N) > c > 0$ for $N \equiv 5 \pmod{24}$.

The proof uses the circle method. The unit interval is in the usual way split into major arcs and minor arcs. The contribution derived from the minor arcs is estimated by the following theorem which is also proved in [6]:

**Theorem 2.** Let $\varepsilon > 0$ be arbitrary, $1 \leq y \leq x$ and
\[
S_2(x, y; \alpha) = \sum_{x < n \leq x + y} A(n)e(n^2\alpha).
\]

Then
\[
S_2(x, y; \alpha) \ll y^{1-\varepsilon} \left( \frac{1}{q} + \frac{x^2}{y} + \frac{x^2}{y^2} + \frac{q x}{y^2} \right)^{\frac{1}{2}}
\]
holds for $\alpha = \frac{a}{q} + \lambda$, $(a, q) = 1$ satisfying $1 \leq q \leq xy$, $|\lambda| \leq \frac{1}{q^2}$.

We will show in this paper that Theorem 1 holds in a weaker form without assuming any hypothesis on the distributions of the zeros of the $L$-functions. More precisely, we will prove:

**Theorem.** There exists a $\delta > 0$ such that every sufficiently large number $N \equiv 5 \pmod{24}$ can be written as
\[
N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,
\]
with $U$ chosen as
\[
U = N^{4-\delta}.
\]

We will adopt a method developed by Liu and Tsang ([4], [5]) to our problem in order to calculate the contribution of the major arcs. Because we follow very closely the work of Liu and Tsang we will often not give all the details of the proof, but refer to the corresponding arguments in [4] and [5]. The minor arcs will be treated by Theorem 2 as in [6].

### 2. Notation and structure of the proof

The most part of our notations will be chosen similar to the notations in [5]. Throughout this paper $p$ always denotes a prime number; $c_1, c_2, \ldots$ are effective positive constants and $\delta$ denotes a small positive number, which will be specified later. $U$ is defined by (1.2) and further let
\[
L = \log N, \quad P = N^{\delta_1}, \quad T = P^{1/\sqrt{\delta_1}}, \quad Q = NT^{-1/4},
\]
where $\delta_1 = 104\delta$. It is a well known fact (see [1]) that there is at most one primitive character to a modulus $q \equiv T$ for which the corresponding $L$-function has a zero in the region
\[
\sigma < 1 - \eta(T), \quad |\epsilon| \leq T, \quad \text{where} \quad \eta(T) = \frac{c_1}{\log T},
\]
for a small constant $c_1$. If there is such an exceptional character, it is real and we denote it by $\tilde{\chi}$. The corresponding exceptional zero is real, simple and unique, and we denote it by $\tilde{\beta}$. If $\tilde{\chi}$ exists, the zero-free region in (2.1) is widened to (see [2])

$$\eta(T) = \frac{c_2}{\log T} \log \left( \frac{ec_1}{(1-\tilde{\beta})\log T} \right).$$

(2.2)

It is further known that for the exceptional modul $\tilde{\chi}$ the estimates

$$\frac{c_3}{\tilde{\rho}^{1/2} \log \tilde{\rho}} \leq 1 - \tilde{\beta} \leq \frac{c_1}{\log T}$$

hold. For any $x > N^{1/4}$ and any $\chi \mod q$ with $q \leq T$ we define:

$$S_{\chi}(x, T) = \sum_{|\gamma| \leq T}^\prime x^{\beta-1},$$

where $\sum_{|\gamma| \leq T}^\prime$ denotes the summation over all zeros $= \beta + i\gamma$ of $L(s, \chi)$ lying inside the region:

$|\gamma| \leq T, \frac{1}{2} \leq \beta \leq 1 - \eta(T)$ and $\eta(T)$ is defined in (2.2) or (2.1) according as $\tilde{\beta}$ exists or not. Let

$$\Omega(T) = \begin{cases} (1 - \tilde{\beta}) \log T, & \text{if } \tilde{\beta} \text{ exists,} \\ 1, & \text{otherwise.} \end{cases}$$

(2.4)

Using these results it can be shown by applying Gallagher’s density estimate ([2]) that the following lemma, which is shown in the same way as Lemma 2.1 in [4], is true.

**Lemma 2.1.** If $x \geq N^{1/4}$ there exists an absolute constant $c_4$ such that for a sufficiently small $\delta_1$,

$$\sum_{q \leq T} \sum_{\chi \mod q}^* S_{\chi}(x, T) \ll \Omega^5(T) \exp \left( -c_4 / \delta_1 \right),$$

where $\sum_{\chi \mod q}^*$ denotes the summation over all primitive characters $\chi \mod q$.

Further for any real $\lambda$ we set $e(\lambda) = e^{2\pi i \lambda}$ and

$$N_1 = \sqrt{\frac{N}{5}} - U, \quad N_2 = \sqrt{\frac{N}{5}} + U,$$

which we use to define

$$S(\alpha) = \sum_{N_1 < n \leq N_2} A(n)e(n^2 \alpha), \quad S_{\chi}(\alpha) = \sum_{N_1 < n \leq N_2} A(n)\chi(n)e(n^2 \alpha),$$

for every character $\chi \mod q$ with $q \leq T$.

$$I(\alpha) = \int_{N_1}^{N_2} e(x^2 \alpha) \, dx, \quad \bar{I}(\alpha) = \int_{N_1}^{N_2} x^{\beta-1} e(x^2 \alpha) \, dx,$$

and

$$I_{\chi}(\alpha) = \int_{N_1}^{N_2} e(x^2 \alpha) \sum_{|\gamma| \leq T}^\prime x^{\beta-1} \, dx.$$
For any character $\chi \mod q$ let
\[
C_\chi(m) = \sum_{l=1}^{q} \chi(l) e\left(\frac{ml^2}{q}\right), \quad C_q(m) = C_{\chi_0}(m).
\]
We write $\sum_{a=1}^{q*} = \sum_{a=1}^{q} (a,q)=1$, recall $Q = NT^{-1/4}$ and define the major arcs and minor arcs as follows:
\[
M = \sum_{q \equiv 1 \mod q} \sum_{a=1}^{q*} I(a,q), \quad I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],
\]
\[
m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.
\]
The major arcs are obviously disjoint subintervals of $\left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$. Writing
\[
I(n) = \sum_{\substack{n_1 < n_2 < \cdots < n_s \leq N \mod q \atop n_1 + \cdots + n_s = N}} A(n_1) \ldots A(n_s),
\]
we obtain
\[
I(N) = \int_{0}^{1+\frac{1}{Q}} e(-na)S^5(\alpha) \, d\alpha = \left(\int_{M} + \int_{m}\right) e(-na)S^5(\alpha) \, d\alpha = I_1(N) + I_2(N).
\]
We will first treat the integral over the major arcs.

3. Simplification of $I_1(N)$. For any $\alpha$ in $I(a,q)$ we have $\alpha = \frac{a}{q} + \eta$ with $|\eta| \leq \frac{1}{qQ}$. In a well known way we obtain
\[
S(\alpha) = \phi^{-1}(q) \sum_{\chi \mod q} C_\chi(a)S_\chi(\eta).
\]
Following the arguments in [4] we will now give four lemmas which we will use to simplify the contribution of the major arcs. Their proofs will not always be given completely because some of them can be shown in exactly the same way as Lemma 3.1. to 3.4. in [4].

Lemma 3.1. For any real $\alpha$ and any $\chi \mod q$ with $q \leq T$, we obtain
\[
S(\eta) = \delta_{\chi_0} I(\eta) - \delta_{\chi} I(\eta) - I_\chi(\eta) + O((1 + |\eta|)NT^{1/2}L^2 T^{-1}),
\]
where
\[
\delta_{\chi_0} = \begin{cases} 1, & \text{if } \chi = \chi_0 \mod q \\ 0, & \text{otherwise} \end{cases}, \quad \delta_{\chi} = \begin{cases} 1, & \text{if } \chi = \chi \chi_0 \mod q \\ 0, & \text{otherwise} \end{cases}
\]
Proof. We note that for $2 \leq T \leq x$ the identity (see [1], p. 109 and p. 120.)
\[
\sum_{n \equiv x} \chi(n)A(n) = \delta_{\chi_0} x - \delta_{\chi} \frac{\chi_0}{\beta} + \sum_{\substack{\beta \mod q \equiv x \beta \mod q \leq T \rho}} \frac{x^\rho}{\beta} + R(x,q)
\]
is valid with $R(x, q) \ll \frac{xL^2}{T} + L$ and the summation is running over all zeros of $L(s, \chi)$ with $0 \leq \text{Re}(\rho) \leq 1, |\text{Im}(\rho)| \leq T$ and the possible Siegel – zero is excluded. Then Lemma 3.1 follows by using partial summation if we note that

$$\int_{N_1}^{N_2} e(x^2 \eta) d(R(x, q)) \ll N^{1/2} L^2 T^{-1} + \int_{N_1}^{N_2} |R(x, q)| \left| \frac{d}{dx} e(x^2 \eta) \right|$$

$$\ll (1 + |\eta|) N^{1/2} L^2 T^{-1}.$$

**Lemma 3.2.** Let $\rho = \beta + i\gamma$, $1/2 \leq \beta \leq 1$. Then for any real $\eta$ it is known that

$$\int_{N_1}^{N_2} e(x^2 \eta) x^{\beta - 1} dx \ll \begin{cases} \min \left( N_2^\beta, |\eta|^{-1/2} N_1^{-1} \right), & \text{if } \gamma = 0, \\ N_2^\beta |\eta|^{-1}, & \text{if } |\gamma| \leq \frac{|\gamma|}{8\pi N_2^2}, \\ N_2^2 N_1^{-2} |\eta|^{-1/2}, & \text{if } \frac{|\gamma|}{8\pi N_2^2} \leq |\eta| \leq \frac{|\gamma|}{2\pi N_1^2}, \\ N_1^{-2} |\eta|^{-1}, & \text{if } \frac{|\gamma|}{2\pi N_1^2} < |\eta|. \end{cases}$$

The proof of this lemma is literally the same as the one of Lemma 3.2 in [4].

**Lemma 3.3.** For any real $\eta$ we obtain

$$I(\eta) \ll \min \left( N_2, |\eta|^{-1} N_1^{-1} \right), \quad \tilde{I}(\eta) \ll \min \left( N_2^\beta, |\eta|^{-1} N_1^{-1} \right),$$

$$L_2(\eta) \ll \begin{cases} N_2, & \text{for any } \eta, \\ N_2^2 N_1^{-2} |\eta|^{-1/2}, & \text{for } N_2^{-2} \leq |\eta| \leq \frac{T}{2\pi N_1^2}, \\ |\eta|^{-1} N_1^{-1}, & \text{for } \frac{T}{2\pi N_1^2} < |\eta|. \end{cases}$$

Using lemma 3.2 this lemma is proved in exactly the same way as Lemma 3.3 in [4].

**Lemma 3.4.**

$$\int_{-\infty}^{\infty} |I(\eta)|^4 d\eta \ll \frac{N_2^4}{N_1^4}, \quad \int_{-\infty}^{\infty} |\tilde{I}(\eta)|^4 d\eta \ll \frac{N_2^{2(\beta + 1)}}{N_1^4},$$

$$\int_{-\infty}^{\infty} |L_2(\eta)|^4 d\eta \ll \frac{N_2^{10}}{N_1^8}.$$

**Proof.** The first estimate follows from Lemma 3.3, if we split up the integral in the following way:

$$\int_{-\infty}^{\infty} |I(\eta)|^4 d\eta \ll \int_{|\eta| \leq N_2^{-2}} N_2^4 d\eta + \int_{N_2^{-2} < |\eta|} |\eta|^{-4} N_1^{-4} d\eta \ll \frac{N_2^6}{N_1^4}.$$
The second estimate is proved in the same way whereas for the proof of the third estimate we split the integral in the following way:

\[
\int_{-\infty}^{\infty} |I_2(\eta)|^4 \, d\eta \ll \int_{|\eta| \leq N_2^{-2}} N_2^2 \, d\eta + \int_{N_2^{-2} < |\eta| \leq \frac{p}{5N^2}} N_2^8 N_1^{-8}(|\eta|)^{-2} \, d\eta \\
+ \int_{\frac{p}{5N^2} < |\eta|} |\eta|^{-4} N_1^{-4} \, d\eta \ll \frac{N_2^2}{N_1^8} + \frac{N_2^{10}}{N_1^8} + \frac{N_2^2}{T^3} \ll \frac{N_2^{10}}{N_1^8}.
\]

We now simplify \(I_1(N)\) in the same way as it is done in [4]. Set

\[
G(a, q, \eta) = \sum_{\chi \mod q} C_{\chi}(a) I_2(\eta)
\]

and

\[
H(a, q, \eta) = C_q(a) I(\eta) - \delta_q C_{\chi(a)} I(\eta) - G(a, q, \eta),
\]

where

\[
\delta_q = \begin{cases} 1, & \text{if } \bar{r}|q, \\ 0, & \text{otherwise.} \end{cases}
\]

For any \(\alpha = \frac{a}{q} + \eta \in I(a, q)\) we obtain by applying Lemma 3.1 to (3.1)

\[
S(\alpha) = \phi^{-1}(q) \left( H(a, q, \eta) + O \left( \sum_{\chi \mod q} (1 + |\eta|N) |C_{\chi}(a)| N^{1/2} L^2 T^{-1} \right) \right).
\]

From the definition of the major arcs we see that \(|\eta|N \leq T^{1/4}\) and trivially we find that

\[
\left| \sum_{\chi \mod q} C_{\chi}(a) \right| \leq \phi^2(q).
\]

So the O-term above is \(\ll \phi(q)N^{1/2} L T^{-3/4}\). Together with the definition of \(I_1(N)\) we obtain

\[
I_1(N) = \sum_{q \leq p} \phi^{-5}(q) \sum_{a=1}^{\phi(q)} e\left( -\frac{a}{q} N \right) \cdot \int_{1/qQ}^{1/qQ} e(-\eta N) H(a, q, \eta) + O(\phi(q)N^{1/2}L^2 T^{-3/4})^5 \, d\eta.
\]

It is easily deduced from Lemma 3.3 that \(H(a, q, \eta) \ll \phi^2(q)N^{1/2}\). Using this relation we see that the grand error term in the last expression for \(I_1(N)\) may be estimated by

\[
\ll \sum_{q \leq p} \phi^{-5}(q) \sum_{a=1}^{\phi(q)} \int_{1/qQ}^{1/qQ} (\phi^2(q)N^{1/2})^4 \phi(q) N^{1/2} L^2 T^{-3/4} \, d\eta \\
\ll N^{3/2} p^{-2} \leq U^4 N^{-1/2} p^{-1}
\]

for a sufficiently small \(\delta_1\) and because of \(\delta_1 = 104\delta\). Hence we reach

\[
I_1(N) = \sum_{q \leq p} \phi^{-5}(q) \sum_{a=1}^{\phi(q)} e\left( -\frac{a}{q} N \right) \int_{1/qQ}^{1/qQ} e(-\eta N) H^2(a, q, \eta) \, d\eta + O(U^4 N^{-1/2} p^{-1}).
\]
The next step will be to extend the range of integration in (3.3) to \((-\infty, \infty)\). The product \(H^5(a, q, \eta)\) is a sum of \((\phi(q) + 2)^5\) terms of the form \(\prod_{j=1}^{5} E_j\), where each \(E_j\) is \(C_0(a)I(\eta), -\delta_q C_\alpha(a)\tilde{T}(\eta)\) or \(-C_\tau(a)I_\tau(\eta)\). We note that for \(|\eta| \geq (qQ)^{-1}\) among the estimates for \(I(\eta), \tilde{I}(\eta)\) and \(I_\tau(\eta)\) in Lemma 3.3 the weakest one is the estimate in the middle range for \(I_\tau(\eta)\). So we obtain

\[
\int_{|\eta| > (qQ)^{-1}} \prod_{j=1}^{5} E_j \, d\eta \ll \phi(q)(qQ)^{1/2} \int_{-\infty}^{\infty} |E_1 E_2 E_3 E_4| \, d\eta.
\]

Because of Lemma 3.4 this is \(\ll \phi(q)(qQ)^{1/2}\). Thus extending the integration to \((-\infty, \infty)\), the total error induced is

\[
\ll \sum_{q \equiv a \mod p} \phi^{-5}(q) \phi(q)(\phi(q) + 2)^2 \phi(q)(qQ)^{1/2} N^{3/2} T^{-1/8} \ll N^{3/2} P^{-2} \leq U^4 N^{-1/2} P^{-1}
\]

for a sufficiently small \(\delta_1\) and because of \(\delta_1 = 104\delta\). So (3.3) can now be written as

\[
I_1(N) = \sum_{q \equiv a \mod p} \phi^{-5}(q) \sum_{a=1}^{q} e\left(-\frac{a}{q}N\right) \int_{-\infty}^{\infty} e(-\eta N) H^5(a, q, \eta) \, d\eta + O(U^4 N^{-1/2} P^{-1}).
\]

4. Final treatment of the major arcs. The following treatment of the major arcs is nearly identical with the procedure in [5]. For the treatment of the singular series we can completely refer to [5]. We recall the definitions \(N_1 = \sqrt{N/5} - U\), \(N_2 = \sqrt{N/5} + U\). We use the following lemma for the calculation of the contribution of the major arcs:

**Lemma 4.1.** For any complex numbers \(\rho_i\) with \(0 < \text{Re}(\rho_i) \leq 1, j = 1, \ldots, 5\), it is known that

\[
\int_{-\infty}^{\infty} e(-N\eta) \prod_{j=1}^{5} \left(\frac{N_j}{N_i}\right) x^{\rho_j-1} e(\eta x^2) \, dx \, d\eta = 2^{-5} N_2^{2j} \prod_{j=1}^{5} (N_j^2 x_j)^{\rho_j-1/2} x_j^{-1/2} \, dx_1 \ldots dx_4,
\]

where

\[
x_5 = N N_2^{-2} - \sum_{j=1}^{4} x_j
\]

and

\[
\mathcal{D} = \{(x_1, \ldots, x_4) : (N_1/N_2)^2 \leq x_1, \ldots, x_5 \leq 1\}.
\]

Furthermore the lower estimate

\[
\int_{\mathcal{D}} \left(\prod_{j=1}^{5} x_j^{-1/2}\right) \, dx_1 \ldots dx_4 \gg U^4 N^{-2}
\]

holds.

**Proof.** (4.1) is shown in exactly the same way as (3.15) in [5]. For the proof of (4.4) we note that because of (4.2) the condition for \(x_5\) in (4.3) is equivalent to

\[
\frac{N}{N_2^2} - 1 \leq \sum_{j=1}^{4} x_j \leq \frac{N - N_2^2}{N_2^2}.
\]
We now define the region \( D_1 \) by
\[
D_1 = \left\{ (x_1, \ldots, x_4) : \frac{(N_1/N_2)^2}{x_1, \ldots, x_4} \leq \frac{N - N_2^2}{4N_2^2} \right\}
\]
and show that it lies in \( D \). Taking into account that \( N/N_2 - 1 < 0 \) we see from (4.3) and (4.5) that the lower bounds of \( D_1 \) are equal to those of \( D \). This together with the relation
\[
\frac{N - N_2^2}{4N_2^2} = \frac{4}{5} N + 2 \sqrt{\frac{N}{5} U - U^2} - \frac{4}{5} N + 8 \sqrt{\frac{N}{5} U + 4U^2}
\]
shows that \( D_1 \) lies in \( D \). Using \( x_j^{-1/2} \leq 1 \) we find that
\[
\int_{D_1} \left( \prod_{j=1}^{5} x_j^{-1/2} \right) dx_1 \ldots dx_4 \geq \int_{D_1} \left( \prod_{j=1}^{4} x_j^{-1/2} \right) dx_1 \ldots dx_4 \geq \left( \frac{N - N_2^2}{4N_2^2} \right)^4 = \left( \frac{10 \sqrt{\frac{N}{5} U - 5U^2} - 4}{4N_2^2} \right)^4 \gg U^4 N^{-2},
\]
which proves (4.4).

We know from the definition of \( H(a, q, \eta) \) that \( H^3(a, q, \eta) \) is a sum of \( 3^5 \) terms which can be divided into three groups:
- \( T_1 \): the term \( (C_q(a)I(\eta))^{5} \),
- \( T_2 \): the 211 terms each of which has at least one \( G(a, q, \eta) \) as factor,
- \( T_3 \): the remaining 31 terms.

We further write for \( i = 1, 2, 3 \)
\[
M_i = \sum_{q \leq p} \phi^{-5}(q) \sum_{a=1}^{q} e\left( -\frac{Na}{q} \right) \int_{-\infty}^{\infty} e(-N\eta) \{ \text{sum of the terms in } T_i \} \, d\eta,
\]
from which we deduce by using (3.4)
\[
(4.6) \qquad I_1(N) = M_1 + M_2 + M_3 + O(U^4 N^{-1/2} P^{-1}).
\]

We also define
\[
(4.7) \qquad \Phi_0 = \frac{N^3}{2^5} \int_{D} \left( \prod_{j=1}^{5} x_j^{-1/2} \right) dx_1 \ldots dx_4,
\]
\[
\sum_{(q)} \chi_1(n_1) \ldots \chi_5(n_5) = \sum_{\begin{subarray}{c} n_1 + \ldots + n_5 = q \ (n_1, q) = 1 \n_1^2 + \ldots + n_5^2 \leq N \end{subarray}} \chi_1(n_1) \ldots \chi_5(n_5),
\]
and
\[
\delta(p) = \begin{cases} \phi^{-5}(2^3)2^3 \sum_{(2^3)} 1 & \text{for } p = 2, \\ \phi^{-5}(p)p \sum_{(p)} 1 & \text{for } p \geq 3. \end{cases}
\]
Without further mentioning it we will make use of the fact that $\prod_{p} s(p) \gg 1$. Finally we know from (4.10) in [5] that
\begin{equation}
\prod_{p \neq \rho} s(p) = \sigma \tilde{\varphi}^{-5}(\sigma \tilde{\varphi}) \sum_{(\sigma \tilde{\varphi})} 1
\end{equation}
holds, where $\sigma = 1, 4$ and 2 for $2 \notin \tilde{\varphi}$, $2 \parallel \tilde{\varphi}$ and $4 \mid \tilde{\varphi}$ respectively. We will now give estimates for the respective contribution of the $M_i$ to $I_i(N)$ from which we can easily calculate the contribution of the major arcs. We first have
\begin{equation}
M_1 = \mathcal{P}_0 \prod_{p} s(p) + O(N^{3/2} P^{-1} \log^{60} P),
\end{equation}
the proof of which is literally the same as the one of Lemma 4.1 in [5]. The next estimates are given by
\begin{equation}
M_3 \ll N_2^2 \tilde{\varphi}^{-1} \log P
\end{equation}
and
\begin{equation}
M_1 + M_3 \gg \Omega^5 \mathcal{P}_0 \prod_{p} s(p) + O(N^{3/2} P^{-1} \log^{60} P).
\end{equation}
(4.10) corresponds to Lemma 4.2 b) in [5]. If $\tilde{\varphi}$ does not exist the term $M_3$ does not appear and (4.11) follows from (4.9) and the definition of $\Omega$. In the other case we follow the proof of Lemma 4.3 in [5] and derive in exactly the same way
\begin{equation}
M_1 + M_3 = \sigma \tilde{\varphi}^{-5}(\sigma \tilde{\varphi}) \prod_{(\sigma \tilde{\varphi})} s(p) \frac{N_3^2}{2^5} \sum_{(\sigma \tilde{\varphi})} \left( \prod_{j=1}^{5} x_j^{-1/2} \right) \times \left( \prod_{j=1}^{5} (1 - \tilde{\varphi}(\tilde{\varphi}_{j}) (N_2^2 x_j) (\tilde{\varphi}^{-1})^{1/2} \right) dx_1 \ldots dx_4 + O(N^{3/2} P^{-1} \log^{60} P).
\end{equation}
Taking into account that for $x_j \in \mathcal{O}$ there is $x_j \equiv N_2^2 \mod N_1$, we obtain
\begin{equation}
\left( \prod_{j=1}^{5} (1 - \tilde{\varphi}(\tilde{\varphi}_{j}) (N_2^2 x_j) (\tilde{\varphi}^{-1})^{1/2} \right) \equiv \prod_{j=1}^{5} (1 - N_1^{\tilde{\varphi}}). \end{equation}
Using the mean value theorem of differential calculus we further obtain
\[ 1 - N_1^{\tilde{\varphi}} \gg (1 - \tilde{\varphi}) \log N_1 \gg (1 - \tilde{\varphi}) \log T = \Omega. \]
Thus we can conclude from (4.12)
\begin{equation}
M_1 + M_3 \gg \Omega^5 \sigma \tilde{\varphi}^{-5}(\sigma \tilde{\varphi}) \prod_{(\sigma \tilde{\varphi})} s(p) \frac{N_3^2}{2^5} \sum_{(\sigma \tilde{\varphi})} \left( \prod_{j=1}^{5} x_j^{-1/2} \right) + O(N^{3/2} P^{-1} \log^{60} P),
\end{equation}
which together with (4.7) and (4.8) proves (4.11). The contribution of $M_2$ is estimated in the same way as the corresponding term in [5]. Thus we reach
\begin{equation}
M_2 \ll \Omega^5 \exp \left( - \frac{c}{\sqrt{\delta_1}} \right) \mathcal{P}_0 \prod_{p} s(p) + O(N^{3/2} P^{-1} \log^{60} p).
\end{equation}
Finally we combine the above estimates and obtain a lower bound for $I_1(N)$. For the error term in (4.9), (4.11) and (4.14) the estimate

$$N^{3/2}P^{-1/2} \log p \ll U^4N^{-1/2}P^{1/2}$$

holds because of $\delta_1 = 104\delta$. We distinguish two cases:

a) $\tilde{r} > P^{1/13}$ or $\tilde{\tilde{r}}$ does not exist. Using (4.6), (4.9), (4.10), (4.14), (4.15) and $\delta_1 = 104\delta$ we obtain for a sufficiently small $\delta_1$

$$I_1(N) \equiv \frac{1}{2} \mathcal{P}_0 \prod_p s(p) + O(U^4N^{-1/2}P^{-1/27}\log P).$$

Finally we derive from (4.4) and (4.7)

$$I_1(N) \gg U^4N^{-1/2}. \tag{4.16}$$

b) $\tilde{r} \leq P^{1/13}$. Using (4.6), (4.11), (4.14) and (4.15) we see

$$I_1(N) \equiv \frac{1}{2} \mathcal{Q} \mathcal{P}_0 \prod_p s(p) + O(U^4N^{-1/2}P^{-1/2}).$$

From (2.3) we conclude

$$\Omega = (1 - \tilde{r}) \log T \equiv c_2 \log T (\tilde{r}^{1/2} \log^{-2} \tilde{r})^{-1} \gg P^{-1/26} \log^{-2} P,$$

from which we deduce

$$I_1(N) \gg U^4N^{-1/2}P^{-1/26} \log^{-10} P. \tag{4.17}$$

5. The minor arcs. Applying (1.1) we obtain

$$\sup_{a \in \mathcal{M}} |S(a)| \ll U^{1+\epsilon} \left( \frac{1}{P} + \frac{N^{1/4}}{U} + \frac{N^{2/3}}{U^2} + \frac{QN^{1/2}}{U^3} \right)^{1/4} \ll U^{1+\epsilon} P^{-1/4}. \tag{5.1}$$

Now we can estimate $I_2(N)$ by

$$\ll \sup_{a \in \mathcal{M}} |S(a)| \int_0^1 |S(a)|^4 \, da \ll U^{1+\epsilon} P^{-1/4} U^{2+\epsilon} \ll U^4N^{-1/2}P^{-3/13},$$

where in the the last step we have used

$$P^{-1/4} \ll U^{1-2\epsilon} N^{-1/2}P^{-3/13}.$$

This is easily seen to be correct because of $104\delta = \delta_1$ and thus

$$P^{1/1} = N^{\frac{1}{2} + \delta_1} = N^{-2\delta} \ll U^{-2\epsilon} N^{-\delta} \ll U^{1-2\epsilon} N^{-1/2}.$$

The theorem follows from (2.6), (4.16), (4.17) and (5.1).

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References


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